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NOT TOO WELL DIFFERENTIABLE LIPSCHITZ ISOMORPHISMS

BY

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ABSTRACT

Several examples of Lipschitz isomorphisms are given with non-surjective derivatives. In particular there is a Lipschitz isomorphism on ℓ_2 that maps a cube affinely into a hyperplane.

Introduction

In this note we give several examples of Lipschitz isomorphisms of an infinite dimensional separable real Hilbert space onto itself whose Gâteaux derivative is not always an isomorphism; we start by analysing a simple example (from [Iv]) of a mapping for which this happens at one point, and obtain at the end an example where this behaviour happens at every point of an arbitrarily given cube in our space. Moreover, the mapping from this example maps the cube affinely into a hyperplane, which shows that even the image of a hyperplane under a Lipschitz isomorphisms need not be Gaussian (or Aronszajn) null. (Since we will not use the notions of null sets here, we refer to [Ar] and [Ph] for the notions and their applications to differentiability, to [Cs] for the equivalence of these notions, to [Bo] for a simple example showing that this notion is not preserved under Lipschitz isomorphisms, and to [Ma] for an example of a Lipschitz isomorphism that maps a set that is not Haar null into a set that is Aronszajn null; this information may also be found in the forthcoming book [BL].)

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Suppose that X and Y are Banach spaces. We recall that a map $f: X \to Y$ is said to be Gâteaux differentiable at x if there is a continuous linear map $f'(x): X \to Y$ such that

$$f'(x)h = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

for all $h \in X$. A map $f: X \to Y$ is a linear isomorphism provided that both f and its inverse have bounded Lipschitz constants.

We start by describing a simple construction of the following example; it will serve as a starting point of our analysis of the construction of not too well differentiable Lipschitz isomorphisms.

Example 1: There is a Lipschitz isomorphism f of ℓ_2 onto itself such that, at some $x \in \ell_2$, the Gâteaux derivative $f'(x)$ exists and is not surjective. ∎

Let us first briefly describe a simple construction of such an example. For every $n = 1, 2, \ldots$ we denote by $x \to R_n(\theta, x)$ the rotation of ℓ_2 through the angle θ about its subspace spanned by the vectors e_k, where $k \neq 1, n$. Define $\theta_n: \ell_2 \to \mathbb{R}$ by $\theta_n(x) = -\pi \min(1/2, \max(0, 1 - 2^n\|x\|))$ and $f_n: \ell_2 \to \ell_2$ by $f_n(x) = R_n(\theta_n(\|x\|), x)$. Then $f = \lim_{n\to\infty} f_n \circ \cdots \circ f_1$ is a Lipschitz isomorphism of ℓ_2 onto itself whose Gâteaux derivative at the origin exists and is given by the shift $f'(0)(\sum h_k e_k) = \sum h_k e_{k+1}$; in particular, it is not surjective.

A direct proof of these statements is straightforward and (in a slightly modified form) it can be found in [Iv] and [BL]. We give the main arguments in the last section, where they will serve as an introduction to the use of the related but somewhat more technical arguments needed to construct other examples.

Our second example is constructed in the same way as the first; we only order the rotations differently.

Example 2: There is a Lipschitz isomorphism f of ℓ_2 onto itself such that, at some $x \in \ell_2$, the weak limit of $(f(x+tu) - f(x))/t$ is zero, as $t \to 0$, for every direction $u \in \ell_2$. ∎

In the third example we construct a Lipschitz isomorphism of ℓ_2 onto itself whose restriction to a particular non-degenerate cube is the shift by one coordinate.

Example 3: There is a Lipschitz isomorphism f of ℓ_2 onto itself such that $f(x) = (0, x_1, x_2, \ldots)$ whenever $x \in \ell_2$ satisfies $|x_j| \leq 2^{-j}$ for each j. ∎

Our last isomorphisms of ℓ_2 have a bad differentiability and mapping behaviour on an arbitrarily given cube; they are, however, obtained by a simple modification of Example 3, which should therefore be considered as the main result of this note.

Example 4: Let α_k be a sequence of positive numbers with $\lim_{k\to\infty}\alpha_k = 0$. Then there is a Lipschitz isomorphism f of ℓ_2 onto itself such that

1. f maps the cube $C = \{x \in \ell_2 \colon |x_j| \le \alpha_j\}$ into a hyperplane,
2. the restriction of f to C is affine, and
3. for every $x \in C$, the Gâteaux derivative $f'(x)$ exists and is not surjective.

∎

We start our constructions by a section pointing out in which way we may construct Lipschitz isomorphisms by a limit procedure. The basic idea is that, although we have to compose arbitrarily large numbers of mappings, we can keep control of the Lipschitz constant if each map has its own part of the space on which it distorts distances, and if on the remaining parts of the space it is composed from isometries. This particular behaviour is clear in the case of the mappings f_n from the above description of the construction of Example 1, since f_n is an isometry on $U_n = \{x \colon \|x\| \le 2^{-n-1}\}$, where it is a rotation, as well as on $\{x \colon \|x\| \ge 2^{-n}\}$, where it is the identity. The behaviour of the derivative at the origin comes from the rotational behaviour of f_n on U_n; this argument works in all our examples. The estimate of the Lipschitz constant of the composition $f_n \circ \cdots \circ f_1$ and of its inverse are enabled by the fact that the barriers $\{x \colon 2^{-n-1} < \|x\| < 2^{-n}\}$ are disjoint. Here, however, the arguments of the more advanced examples become somewhat less straightforward, and we therefore start with a section devoted to developing the technique in some generality; we begin by collecting the basic methods, and continue by improving them to a technical form suitable for our purpose. The constructions of the examples are deferred to the last section.

Constructions of Lipschitz isomorphisms

LEMMA 1:

(a) *If X, Y, Z are metric spaces, $g_n \colon X \to Y$, $h_n \colon Y \to Z$, g_n converge pointwise to g, h_n converge pointwise to h, and $\mathrm{Lip}(h_n) \le L < \infty$, then $h_n \circ g_n$ converge pointwise to $h \circ g$.*

(b) *If $g_n \colon X \to Y$ are Lipschitz isomorphisms such that g_n converge pointwise to g, g_n^{-1} converge pointwise to h, and $\mathrm{Lip}(g_n), \mathrm{Lip}(g_n^{-1}) \le L < \infty$, then g is a Lipschitz isomorphism of X onto Y.*

Proof: The statement (a) follows from

$$\mathrm{dist}(h_n(g_n(x)), h(g(x))) \le L\,\mathrm{dist}(g_n(x), g(x)) + \mathrm{dist}(h_n(g(x)), h(g(x))) \to 0,$$

and the statement (b) follows by deducing from (a) that $h \circ g$ is the identity on X, $g \circ h$ is the identity on Y, and noting that a pointwise limit of a sequence of functions with uniform bound on their Lipschitz constants is Lipschitz. ∎

We remark that the above argument generalises the well known fact that, if f is a Lipschitz isomorphism between Banach spaces X and Y such that, for some $a \in X$, the Gâteaux derivatives of f at a and of f^{-1} at $b = f(a)$ exists, then $f'(a)$ is a linear isomorphism of X onto Y. Indeed, consider $g_n(x) = n(f(a + x/n) - f(a))$ and $h_n(y) = n(f^{-1}(b + y/n) - f^{-1}(b))$. By definition of the Gâteaux derivative, $g_n(x) \to f'(a)(x)$ and $h_n(y) \to f^{-1\prime}(b)(y)$; since $g_n \circ h_n$ and $h_n \circ g_n$ are identities, we infer that $f'(a) \circ f^{-1\prime}(b)$ and $f^{-1\prime}(b) \circ f'(a)$ are identities, so $f'(a)$ is a linear isomorphism of X onto Y.

We will use the following Lemma only for finite covers, and, by a simple modification of our construction, we could use it only for finite open covers, in which case its proof would become even simpler.

LEMMA 2: *If C is a convex set in a normed linear space X, Y is a metric space, $f\colon C \to Y$ is continuous and C can be covered by countably many sets on each of which the Lipschitz constant of f does not exceed L, then* $\mathrm{Lip}(f) \le L$.

Proof: It suffices to consider the case when $C = [a,b] \subset \mathbb{R}$ and to show that $\mathrm{dist}(f(b), f(a)) \le L(b-a)$. Suppose that $\mathrm{dist}(f(a), f(b)) > L(b-a)$. Let $[a,b] = \bigcup_{i=1}^{\infty} M_i$, where M_i are sets on which the Lipschitz constant of f does not exceed L. Let $S = \{\sup(M_i)\colon i = 1, 2, \ldots\}$. The function $g(t) = \mathrm{dist}(f(a), f(t)) - L(t-a)$ is continuous on $[a,b]$ and $g(a) = 0 < g(b)$. Using that $g(S)$ is countable, we choose $c \in [g(a), g(b)] \setminus g(S)$ and use the intermediate value theorem to find the last $t \in [a,b]$ such that $g(t) = c$. Whenever $t < s \le b$, then $g(s) > g(t)$, which gives $\mathrm{dist}(f(s), f(t)) \ge \mathrm{dist}(f(a), f(s)) - \mathrm{dist}(f(a), f(t)) \ge g(s) - g(t) + L(s-t) > L(s-t)$. Finding M_i containing t, we infer that t is the maximum of M_i, so $t \in S$, which contradicts $g(t) = c \notin g(S)$. ∎

LEMMA 3: *Suppose that $h_1, \ldots, h_n$ are Lipschitz mappings of a Banach space X onto itself and that for each k there is a set $A_k \subset X$ such that*

1. *the restriction of h_k to A_k has Lipschitz constant at most one,*
2. *$h_k(X \setminus A_k) \subset X \setminus A_{k+1}$ whenever $k < n$, and*
3. *the restriction of h_{k+1} to $h_k(X \setminus A_k)$ has Lipschitz constant at most one whenever $k < n$.*

Then

$$\mathrm{Lip}(h_n \circ \cdots \circ h_1) \le \max(\mathrm{Lip}(h_n), \ldots, \mathrm{Lip}(h_1)).$$

Proof: Let g_0 be the identity and $g_k = h_k \circ \cdots \circ h_1$ for $k \geq 1$. Note that, whenever $g_{j-1}(x) \in X \setminus A_j$ for some j, then the second assumption implies that $g_{k-1}(x) \in X \setminus A_k$ for all $k \geq j$. Hence the sets

$$M_j = \bigcap_{k=1}^{j-1} g_{k-1}^{-1}(A_k) \cap \bigcap_{k=j}^{n} g_{k-1}^{-1}(X \setminus A_k),$$

where $j = 1, \ldots, n+1$ (with $\bigcap_{k=1}^{0} g_{k-1}^{-1}(A_k) = X$ and $\bigcap_{k=n+1}^{n} g_{k-1}^{-1}(X \setminus A_k) = X$) cover X.

The restriction of g_n to each such M_j is a composition of the restriction of h_1 to $A_1, \ldots, h_{j-1}$ to A_{j-1}, which all have Lipschitz constant at most one according to the first assumption, followed by h_j whose Lipschitz constant we estimate by $\mathrm{Lip}(h_j)$, and followed by the restriction of h_{j+1} to $h_j(X \setminus A_j), \ldots, h_n$ to $h_{n-1}(X \setminus A_{n-1})$, which all have Lipschitz constant at most one according to the last assumption. Hence the restriction of g_n to each M_j has Lipschitz constant at most $\max(\mathrm{Lip}(h_n), \ldots, \mathrm{Lip}(h_1))$. Since g_n is continuous (it is even Lipschitz), by Lemma 2 it has Lipschitz constant at most $\max(\mathrm{Lip}(h_n), \ldots, \mathrm{Lip}(h_1))$. ∎

LEMMA 4: *Suppose that g_k are Lipschitz isomorphisms of a Banach space X onto itself such that $\mathrm{Lip}(g_k), \mathrm{Lip}(g_k^{-1}) \leq L$ for some $L < \infty$ and that for each k there is a non-empty set $U_k \subset X$ such that*

1. *the restriction of g_k to U_k is an isometry,*
2. *the map g_k maps U_k onto itself,*
3. $U_k \supset U_{k+1}$,
4. $g_{k+1}(x) = x$ *for every* $x \in X \setminus U_k$, *and*
5. $\lim_{k\to\infty} \sup_{x\in X} \mathrm{dist}(x, X \setminus U_k) = 0$.

Then the (uniform) limit

$$f = \lim_{n\to\infty} g_n \circ \cdots \circ g_1$$

exists and defines a Lipschitz isomorphism of X onto itself.

Proof: For each n, we use Lemma 3 with $h_k = g_k$ and $A_k = U_k$ to infer that the Lipschitz constant of $f_n = g_n \circ \cdots \circ g_1$ does not exceed L; the assumptions of the Lemma are satisfied since the restriction of g_k to U_k is an isometry by (1), $g_k(X \setminus U_k) = X \setminus U_k \subset X \setminus U_{k+1}$ by (2) and (3), and the restriction of g_{k+1} to $g_k(X \setminus U_k) = X \setminus U_k$ is an isometry by (4). Then we use Lemma 3 again with $h_k = g_{n-k+1}^{-1}$, $A_k = X \setminus U_{n-k}$ for $k < n$ and $A_n = \emptyset$ to infer that the Lipschitz constant of $f_n^{-1} = g_1^{-1} \circ \cdots \circ g_n^{-1}$ also does not exceed L; the assumptions of the Lemma are satisfied since by (4) the restriction of g_{n-k+1}^{-1} to $X \setminus U_{n-k}$ is the identity, so an isometry and the set U_{n-k} is mapped by g_{n-k+1}^{-1} onto itself, which together with (3) can be used to show that $g_{n-k+1}^{-1}(X \setminus A_k) = U_{n-k} \subset U_{n-k-1} = X \setminus A_{k+1}$ and, together with (1), that the restriction of g_{n-k}^{-1} to $g_{n-k+1}^{-1}(X \setminus A_k) = g_{n-k+1}^{-1}(U_{n-k}) = U_{n-k}$ is an isometry.

Given $\varepsilon > 0$ (using (5)) we find n such that for any $x \in X$ there is $z \in X \setminus U_n$ such that $\|z - x\| < \varepsilon$. From (3) and (4) we infer that

$$g_m \circ g_{m-1} \circ \cdots \circ g_{n+1}(z) = z$$

for $m > n$. Applying f_m^{-1}, we get that

$$f_n^{-1}(z) = f_m^{-1}(z),$$

so

$$\begin{aligned}\|f_m^{-1}(x) - f_n^{-1}(x)\| &\le \|f_m^{-1}(x) - f_m^{-1}(z)\| + \|f_m^{-1}(z) - f_n^{-1}(x)\| \\ &= \|f_m^{-1}(x) - f_m^{-1}(z)\| + \|f_n^{-1}(z) - f_n^{-1}(x)\| \\ &\le 2L\|z - x\| \\ &< 2L\varepsilon,\end{aligned}$$

and we see that the sequence f_n^{-1} is uniformly convergent. Similarly, choosing $z \in X \setminus U_n$ such that $\|z - f_n(x)\| < \varepsilon$, and letting $y = f_n^{-1}(z)$, we infer from (3) and (4) that $f_n(y) = f_m(y)$ for $m > n$, and we estimate

$$\begin{aligned}\|f_m(x) - f_n(x)\| &\le \|f_m(x) - f_m(y)\| + \|f_m(y) - f_n(x)\| \\ &= \|f_m(x) - f_m(y)\| + \|f_n(y) - f_n(x)\| \\ &\le 2L\|y - x\| = 2L\|f_n^{-1}(z) - f_n^{-1}(f_n(x))\| \\ &\le 2L^2\|z - f_n(x)\| \\ &< 2L^2\varepsilon,\end{aligned}$$

which gives that the sequence f_n is uniformly convergent. Consequently, our statement follows from Lemma 1. ∎

LEMMA 5: *Suppose that $c > 0$, V is a two dimensional subspace of a Hilbert space H with orthonormal basis v_1, v_2 and orthogonal complement W, and that U is a non-empty subset of H which is mapped onto itself by rotations of H about W (i.e., whenever $v, v' \in V$ have the same norm and $w \in W$, then $v + w \in U$ if and only if $v' + w \in U$).*

Then there is a Lipschitz isomorphism $g\colon H \to H$ such that

1. *$g(u) = u$ whenever $\operatorname{dist}(u, U) \ge c \sup\{\|z\|: z \in V,\ w \in W,\ z + w \in U\}$,*

2. *$g(xv_1 + yv_2 + w) = yv_1 - xv_2 + w$ whenever $x, y \in \mathbb{R}$ and $w \in W$ are such that $xv_1 + yv_2 + w \in U$,*
3. *$g(u)$ is obtained from u by a rotation about W,*
4. *the map g maps the set U onto itself, and*
5. $\mathrm{Lip}(g), \mathrm{Lip}(g^{-1}) \le \kappa$, *where κ is a constant depending on c only.*

Proof: Let $r = \sup\{\|z\|: z \in V,\ w \in W,\ z + w \in U\}$; we may clearly assume that $0 < r < \infty$ and define $\theta: H \to \mathbb{R}$ by

$$\theta(u) = \max(0, 1 - \mathrm{dist}(u, U)/cr).$$

For notational purposes, it will be convenient to identify, for the rest of this proof, the point $xv_1 + yv_2 \in V$ with the complex number $z = x + iy \in \mathbb{C}$; thus H becomes identified with $\mathbb{C} \oplus W$. (So, $\mathbb{C}$ is still considered as a two dimensional real Hilbert space.) For $\sigma = \pm 1$ let $h_\sigma: \mathbb{C} \oplus W \to \mathbb{C} \oplus W$ be defined by

$$h_\sigma(u) = \exp(-i\pi\sigma\theta(u)/2)z + w \quad \text{if } u = z + w, \quad z \in \mathbb{C}, \quad w \in W.$$

We show that $g = h_1$ has the required properties. To see (1), it suffices to note that $\theta(u) = 0$ for $u \in N = \{u: \mathrm{dist}(u, U) \ge cr\}$, so $h_\sigma(u) = u$ for all $u \in N$. For (2), we observe that $\theta(u) = 1$ for $u \in U$, so if $u = z + w, z \in \mathbb{C}, w \in W$, then $h_1(u) = -iz + w$. Clearly $h_\sigma(u)$ is obtained from u by a rotation about W, which shows (3). Since U is rotationally invariant, (4) follows.

Note that the function $u \to \mathrm{dist}(u, U)$ is invariant under rotations about W. Since $h_\sigma(u)$ is obtained from u by a rotation about W, it follows that $\mathrm{dist}(u, U) = \mathrm{dist}(h_\sigma(u), U)$, which, according to the definition of θ, implies $\theta(u) = \theta(h_\sigma(u))$. Consequently,

$$h_{-\sigma}(h_\sigma(u)) = \exp(i\pi\sigma\theta(h_\sigma(u))/2)\exp(-i\pi\sigma\theta(u)/2)z + w = z + w$$

whenever $u = z + w, z \in \mathbb{C}, w \in W$. So h_σ are bijections, and $h_\sigma^{-1} = h_{-\sigma}$.

It remains to prove the required estimate of the Lipschitz constant of h_σ. To obtain it, we first note that the mapping $\psi: [-1, 1] \times \{z \in \mathbb{C}; |z| \le 1 + 1/c\} \to \mathbb{C}$ defined by $\psi(s, z) = \exp(-i\pi s/2)z$ is continuously differentiable, so its Lipschitz constant is bounded by a constant $\kappa \ge 1$ which depends on c only. We first show that the Lipschitz constant of the restriction of h_σ to the cylinder

$$B = \{z + w: z \in V,\ \|z\| \le (1 + c)r \text{ and } w \in W\}$$

is at most κ. Since the W component of h_σ has Lipschitz constant one, it suffices to consider its $\mathbb{C}$ component, say η, which can be written as

$$\eta = cr\psi \circ \phi$$

where $\phi\colon B \to [-1,1] \times \{z \in \mathbb{C}\colon |z| \le 1 + 1/c\}$ is defined by

$$\phi(u) = (\sigma\theta(u), z/cr) \quad \text{if } u = z + w, \quad z \in \mathbb{C}, \quad w \in W.$$

Since θ has Lipschitz constant $1/cr$, we have that $\mathrm{Lip}(\phi) \le 1/cr$, and infer that $\mathrm{Lip}(\eta) \le \kappa$ as required. Finally, we note that $\theta(u) = 0$ for $u \notin B$, so h_σ is the identity on the complement of B; since h_σ is continuous, Lemma 2 implies that $\mathrm{Lip}(h_\sigma) \le \kappa$. ∎

1. Construction of the examples

In this section we denote by e_n the standard orthonormal basis of ℓ_2.

CONSTRUCTION OF EXAMPLE 1: The construction from the introduction may now be described by letting $U_n = \{x \in \ell_2\colon \|x\| \le 2^{-n}\}$ and by using Lemma 5 with $c = 1$, $v_1 = v_1^{(n)} = e_1$ and $v_2 = v_2^{(n)} = e_{n+1}$ to find Lipschitz isomorphisms $f_n\colon \ell_2 \to \ell_2$ with uniformly bounded Lipschitz constants such that

1. $f_n(x) = x$ whenever $\|x\| \ge 2^{-n+1}$,
2. $f_n(x) = (x_{n+1}, x_2, \dots, x_n, -x_1, x_{n+2}, x_{n+3}, \dots)$ for every $x \in U_n$,
3. $f_n(u)$ is obtained from u by a rotation about $\{x \in \ell_2\colon x_1 = x_{n+1} = 0\}$, and
4. the map f_n maps the set U_n onto itself.

Then the assumptions of Lemma 4 are clearly satisfied, so the limit

$$f = -\lim_{n\to\infty} f_n \circ \cdots \circ f_1$$

defines a Lipschitz isomorphism of ℓ_2 onto itself.

To show that $f'(0)$ is not surjective, we note that the mappings f_n preserve the norm, so all sets $f_n(U_k) = U_k$ for all k, n, and

$$f_n \circ \cdots \circ f_1(x) = (x_{n+1}, -x_1, -x_2, \dots, -x_n, x_{n+2}, x_{n+3}, \dots)$$

for $x \in U_n$. Whenever $u \in \ell_2$ is such that $u_k = 0$ for $k \ge m$, we infer that for $|t|$ small enough,

$$f_m \circ \cdots \circ f_1(tu) = (0, -tu_1, -tu_2, \dots, -tu_m, 0, 0, \dots);$$

since by (3) this point is fixed by all f_k with $k > m$, we conclude that $f(tu) = (0, tu_1, tu_2, \dots)$ for $|t|$ sufficiently small. Since f is Lipschitz, this shows that $f'(0)(u) = (0, u_1, u_2, \dots)$ for all $u \in \ell_2$.

It would not be difficult in this example to have that f is everywhere Gâteaux differentiable simply by replacing θ with a smooth function given by

$$\theta(u) = \begin{cases} 1 & \text{if } \|u\| \le 2^{-n}, \\ \sin^2(2^{n-1}\pi\|u\|) & \text{if } 2^{-n} \le \|u\| \le 2^{-n+1}, \\ 0 & \text{if } \|u\| \ge 2^{-n+1}. \end{cases}$$

Construction of Example 2: We use the same construction as for Example 1, the only difference being that to construct f_n we use the orthonormal vectors $v_1^{(n)} = e_{k+1}$ and $v_2^{(n)} = e_{k+1+2^m}$, where $n = 2^m + k$, $m \ge 0$, $0 \le k < 2^m$.

The proof that the limit $f = \lim_{n\to\infty} f_n \circ \cdots \circ f_1$ defines a Lipschitz isomorphism of ℓ_2 onto itself is the same as above.

We show that the weak derivative of f at the origin is zero; since f is Lipschitz, this would follow once we show (a stronger property) that if all non-zero coordinates of u occur before some 2^p, then the first $2^p - 1$ coordinates of $f(tu)$ are zero provided that $|t|$ is small enough: since by (2) (with e_{k+1} in place of e_1 and e_{k+1+2^m} in place of e_{n+1}) the mappings f_n for $n < 2^p$ can change only the first $2^p - 1$ coordinates, all coordinates beyond the first $2^p - 1$ of $f_{2^p-1} \circ \cdots \circ f_1(tu)$ are still zero. If $|t|$ is small enough, a direct calculation then shows that $f_{2^{p+1}-1} \circ \cdots \circ f_1(tu)$ has zero as the first $2^p - 1$ coordinates as well as all coordinates from 2^{p+1} on. Using (3) we then see by induction that, if $m \ge p+1$, then $f_{2^m} \circ \cdots \circ f_1(tu)$ can have non-zero coordinates only for indices $2^p \le j < 2^m$, which shows that the first $2^p - 1$ coordinates of $f(tu)$ are zero. ∎

Construction of Example 3: We define $U_1 = \ell_2$ and for $k \ge 2$

$$U_k = \{x \in \ell_2 \colon x_1^2 + x_{k+1}^2 \le 2^{-2k+4} \text{ and } |x_j| \le 2^{-j+1} + 2^{-k} \text{ for } 2 \le j \le k\}.$$

Using Lemma 5 with $c = 1/4$, $v_1 = v_1^{(k)} = e_1$ and $v_2 = v_2^{(k)} = e_{k+1}$ we find Lipschitz isomorphisms $g_k \colon \ell_2 \to \ell_2$ with uniformly bounded Lipschitz constants such that

1. $g_k(x) = x$ whenever $\operatorname{dist}(x, U_k) \ge 2^{-k}$,
2. $g_k(x) = (x_{k+1}, x_2, \ldots, x_k, -x_1, x_{k+2}, x_{k+3}, \ldots)$ for every $x \in U_k$, and
3. the set U_k is mapped by g_k onto itself.

We show that, if $x \in \ell_2 \setminus U_k$, then $\operatorname{dist}(x, U_{k+1}) \ge 2^{-k-1}$. If $x \in \ell_2 \setminus U_k$, then $x_1^2 + x_{k+1}^2 > 2^{-2k+4}$ or $|x_j| > 2^{-j+1} + 2^{-k}$ for some $2 \le j \le k$. Let $y \in U_{k+1}$; then $y_1^2 + y_{k+2}^2 \le 2^{-2k+2}$ and $|y_j| \le 2^{-j+1} + 2^{-k}$ for each $2 \le j \le k+1$. We estimate that

$$y_1^2 + y_{k+1}^2 \le 2^{-2k+2} + 2^{-2k+1} + 2^{-2k-2} = 5^2.2^{-2k-2} \tag{1}$$

so that if $x_1^2 + x_{k+1}^2 > 2^{-2k+4}$ then

$$\|x - y\| > 2^{-k+2} - (5^2.2^{-2k-2})^{1/2} = 3.2^{-k-1} > 2^{-k-1}.$$

If $|x_j| > 2^{-j+1} + 2^{-k}$ then

$$\|x - y\| \geq |x_j| - |y_j| > 2^{-k} - 2^{-k-1} = 2^{-k-1}$$

so that in both cases $\|x - y\| \geq 2^{-k-1}$, and we infer that $\operatorname{dist}(x, U_{k+1}) \geq 2^{-k-1}$. This inequality shows that $U_k \supset U_{k+1}$ and $g_{k+1}(x) = x$ for $x \in U_k$.

Since all the other assumptions of Lemma 4 are obvious, we may use it to infer that the limit

$$f = -\lim_{k\to\infty} g_k \circ \cdots \circ g_1$$

defines a Lipschitz isomorphism of ℓ_2 onto itself.

Let $C_1 = \{x \in \ell_2\colon |x_j| \leq 2^{-j} \text{ for all } j\}$ and, for $k \geq 2$, let

$$C_k = \{x \in \ell_2\colon |x_1| \leq 2^{-k}, |x_j| \leq 2^{-j+1} \text{ for } 2 \leq j \leq k, \text{and } |x_j| \leq 2^{-j} \text{ for } j > k\}.$$

Then $C_k \subset U_k$, so the expression for g_k on U_k gives that $g_k(C_k) = C_{k+1}$. We infer that for every $x \in C_1$,

$$-g_k \circ \cdots \circ g_1(x) = (-x_{k+1}, x_1, x_2, \ldots, x_k, -x_{k+2}, -x_{k+3}, \ldots),$$

which in the limit as $k \to \infty$ shows that $f(x) = (0, x_1, x_2, \ldots)$. ∎

Construction of Example 4: Let $n_1 < n_2 < \cdots$ be such that $\alpha_{n_1} < 1$ and $\alpha_{n_{k+1}} < \alpha_{n_k}/2$. Denote by V the span of $\{e_{n_j}\colon j = 1, 2, \ldots\}$, by W its orthogonal complement, and use Example 3 to find a Lipschitz isomorphism g of V onto itself such that $g(\sum_{j=1}^{\infty} x_{n_j} e_{n_j}) = \sum_{j=1}^{\infty} x_{n_j} e_{n_{j+1}}$ for every $x \in V$ such that $|x_{n_j}| \leq 2^{-j}$ for $j = 1, 2, \ldots$.

Define $f(v + w) = g(v) + w$ whenever $v \in V$ and $w \in W$. Then the image of the set

$$D = \{x \in \ell_2\colon |x_{n_j}| < 2^{-j} \text{ for } j = 1, 2, \ldots\}$$

lies in the hyperplane $\{x \in \ell_2\colon x_{n_1} = 0\}$. To show the statement concerning the non-surjectivity of the derivative, we show that $f'(x)(w + \sum_{j=1}^{\infty} v_{n_j} e_{n_j}) = w + \sum_{j=1}^{\infty} x_{n_j} e_{n_{j+1}}$ for every $x \in D$ and every $u = w + \sum_{j=1}^{\infty} v_{n_j} e_{n_j}$. Since f is Lipschitz, it suffices to verify this only when u has finitely many non-zero coordinates, in which case it is clear since for $|t|$ sufficiently small we have $x + tu \in D$, so $(f(x+tu) - f(x))/t = w + \sum_{j=1}^{\infty} x_{n_j} e_{n_{j+1}}$. The statements of the example follow since $C \subset D$. ∎

References

[Ar] N. Aronszajn, *Differentiability of Lipschitzian mappings between Banach spaces*, Studia Mathematica **57** (1976), 147–190.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, book in preparation.

[Bo] V. I. Bogachev, *Some results on differentiable measures*, Mathematics of the USSR-Sbornik **55** (1986), 335–349.

[Cs] M. Csörnyei, *Aronszajn null and Guassian null sets coincide*, Israel Journal of Mathematics (1999), to appear.

[Iv] D. J. Ives, *PhD Thesis*, University of London, 1998.

[Ma] E. Matoušková, *Lipschitz images of Haar null sets*, to appear.

[Ph] R. R. Phelps, *Gaussian null sets and differentiability of Lipschitz maps on Banach spaces*, Pacific Journal of Mathematics **77** (1978), 523–531.